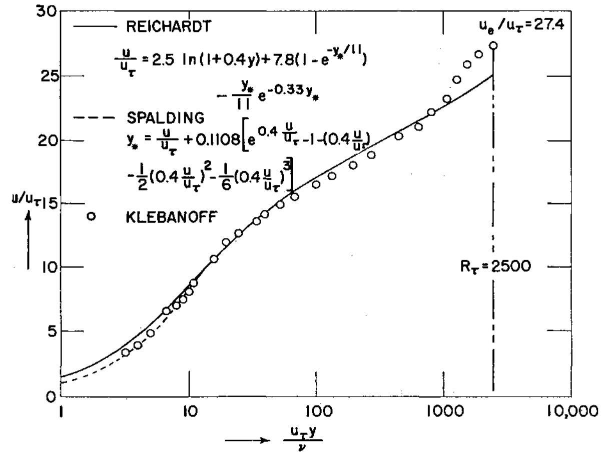

Fig. 3 Velocity profiles at $R_T = 2500$.

A similar comparison, plotted in different coordinates, with the experimental data of Klebanoff as presented by Coles in Ref. 7 is shown in Fig. 3. Although it is not shown here, the velocity profiles derived from the present theory are also compared with the experimental profiles at different R_τ and u_e/u_τ and are proved to be quite satisfactory at a wide range of flow conditions.

It must be noted that the turbulent boundary-layer problem can, in the present state of knowledge, be attacked only by semiempirical method. The extent to which the analysis is simplified depends on how far it appears possible to derive or interpret, in an empirical manner, general relations for the unknown quantities from available experimental data. Therefore, it is not surprising that the various investigators have quite different viewpoints with regards to the simplification of the problem. In this connection, one may attempt to simplify the present approach somewhat further by neglecting the small portion of contribution to the momentum integral from the velocity profile of Eq. (1). That is, Eq. (2) may be used throughout the entire region of the boundary layer as long as the integral approximation is applied for the solution of the gross quantities (such as the momentum thickness). The boundary conditions at the wall, however, must be evaluated from Eq. (1). In conclusion, it may be stated that the single formula for the velocity profile derived in the present analysis provides a relatively simple, consistent, and fairly accurate means of computing the quantitative descriptions of the turbulent boundary layer.

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Stepped Circular Kirchhoff Plate

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THE method of initial parameters is becoming increasingly popular as an approach for solving the deformation problems of classical advanced strength of materials.[1] Relative to many familiar solution schemes, the century-old method of initial parameters involves algebraically complicated expressions, a fact that dampened enthusiasm for the approach until the widespread acceptance of the digital computer as an engineering tool. The method is characterized by the favorable properties of broad scope in covering numerous problems, ease in applicability to problems with complicated loadings or boundary conditions, and appeal to engineers wishing to avoid complex mathematical manipulations and preferring to retain visible control of the physics of the problem.

The present work is intended to contribute the axially symmetric stepped circular plate to the growing catalog of initial parameter solutions. Solutions have already been established for most conceivable forms of the Euler-Bernoulli beams[2] and for some circular plates, rectangular plates, cylinders, rotationally symmetric shells, and arches. In addition, these fundamental solutions for structural members have been extended in bona fide initial parameter form to the multiple statically indeterminate case of an unlimited number of intermediate conditions, including discrete generalized elastic springs or rigid supports (that is, continuous beams in the case of beams). Much of this work reached a well-developed state more than a score of years ago in the Soviet Union.[3] Today, solutions in this field appear in German and British literature under such titles as "transfer matrices" and extensions of "Macauley's Method," respectively.

The axially symmetric flexure of a circular plate of stepped cross section can be handled with classical methods[4,5] by matching appropriate physical variables at each side of each step of a sectionally applied solution (such as the initial parameter solution[6]) for the plate of constant cross section. However, the current work yields for a plate with unlimited steps a true initial parameter solution that can easily be programmed for a digital computer. Advantage can be taken of the usual benefits of initial parameter solutions such as the fully tabulated values (in terms of the initial parameter functions and loading functions that vary from member to member) of the initial parameters themselves.

The fundamental equations describing axially symmetric bending motion of a circular plate that is subject to the assumptions of Kirchhoff's plate theory are given by[7] 1) the loading intensity in force per unit area

$$q(r) = \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{D}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} - \frac{1}{r}\frac{d}{dr}\left[r(1-\mu)\frac{dD}{dr}\frac{1}{r}\frac{dw}{dr}\right] \quad (1)$$

2) the total shearing force on a cylindrical section of radius r

$$V(r) = 2\pi r\left\{\frac{d}{dr}\left[D\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] - (1-\mu)\frac{dD}{dr}\left(\frac{1}{r}\frac{dw}{dr}\right)\right\}$$

3) the total radial moment on a cylindrical section of radius r

$$M(r) = -2\pi r D\,[(d^2w/dr^2) + (\mu/r)(dw/dr)]$$

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4) the slope $\theta(r) = dw/dr$, and 5) the deflection $w(r)$. The usual sign conventions and notation have been employed.

The initial parameter solution, or the universal equations, so called because of their generality in embracing arbitrary boundary and loading conditions with virtual indifference as to complexity, can be written (Fig. 1)

$$\zeta_i(r) = w_0\zeta_{i1}(r, b_0) + \theta_0\zeta_{i2}(r, b_0) + M_0\zeta_{i3}(r, b_0) + V_0\zeta_{i4}(r, b_0) + \psi_i(r) \qquad (i = 1, 2, 3, 4) \quad (2)$$

where

$$\zeta_i(r) = \begin{cases} w(r) & i = 1 \\ \theta(r) & i = 2 \\ M(r) & i = 3 \\ V(r) & i = 4 \end{cases} \qquad \zeta_{ik}(r, a_j) = \begin{cases} w_k(r, a_j) & i = 1 \\ \theta_k(r, a_j) & i = 2 \\ M_k(r, a_j) & i = 3 \\ V_k(r, a_j) & i = 4 \end{cases}$$

$$\psi_i(r) = \begin{cases} W(r) & i = 1 \\ \Theta(r) & i = 2 \\ m(r) & i = 3 \\ v(r) & i = 4 \end{cases}$$

Also, w_0, θ_0, M_0, and V_0 are the initial parameters, that is, the value of the deflection, slope, moment, and reaction, respectively, at the inner boundary $r = a_0 = b_0$; $w_k(r, a_j)$, $\theta_k(r, a_j)$, $M_k(r, a_j)$, and $V_k(r, a_j)$ are the initial parameter functions; $W(r)$, $\Theta(r)$, $m(r)$, and $v(r)$ are the particular solutions, perhaps appropriately called loading functions. The initial parameters have long since been recorded for all common boundary conditions. Moreover, once the initial parameter functions for the stepped circular plate have been derived, the scope of applicability can be broadened by inserting them in the already-established initial parameter functions for multiple statically indeterminate structural members.

The initial parameter functions can be acquired in a straightforward manner, since Eq. (1) has been written such that it can be directly integrated if D is stepped and expressed with unit step functions. Suppose D involves n sudden changes, then

$$\frac{1}{D(r)} = \frac{1}{D_j} + \sum_{i=j+1}^{n} \alpha_i H(r, b_i)$$

$$D(r) = D_j + \sum_{i=j+1}^{n} \gamma_i H(r, b_i)$$

where j plays the very special role (throughout the paper) of being the index for the letters a or b which denote the location of the initiation of a loading or D at the loading, respectively, as shown in Fig. 1 (at $j = 0$, $a_j = a_0 = b_0$). Also, $H(r, b_i)$ is the unit step function

$$\alpha_i = 1/D_i - 1/D_{i-1} \qquad \gamma_i = D_i - D_{i-1}$$

Then, using well-known properties of singularity functions and the identity

$$\left(\frac{1}{D_j} + \sum_{i=j+1}^{n} \alpha_i H(r, b_i)\right) \sum_{k=j+1}^{n} \gamma_k H(r, b_k) = \sum_{k=j+1}^{n} H(r, b_k)\left(\gamma_k \sum_{i=j}^{k} \alpha_i + \alpha_k \sum_{i=j+1}^{k-1} \gamma_i\right) \quad (3)$$

with $\alpha_j = 1/D_j$, integration of Eq. (1) yields

$$w(r = \sum_{i=0}^{n} H(r, b_i)\, \alpha_i \left\{\int_{b_i}^{r} \frac{1}{x} \int_{b_i}^{x} u \int^{u} \frac{1}{t} \int^{t} q(s) \times s\, ds\, dt\, du\, dx + \frac{1}{2}(\beta_i + C_1\alpha_i) \times \left(\frac{r^2 - b_i^2}{2} - b_i^2 \ln \frac{r}{b_i}\right)\right\} + C_2 \ln \frac{r}{b_0} + C_3 \quad (4)$$

where C_k are arbitrary constants of integration, $\beta_0 = 0$, otherwise

$$\beta_i = (1 - \mu)\left\{\frac{dw}{dr}(b_i)\frac{\gamma_i}{b_iD_i} + \alpha_i \sum_{k=1}^{i-1} \frac{\gamma_k}{b_k}\frac{dw}{dr}(b_k)\right\}$$

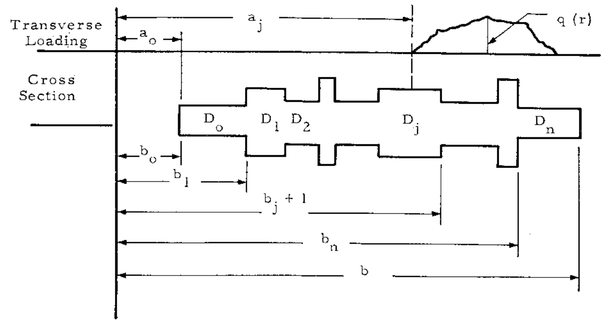

Fig. 1 Plate with stepped cross section and arbitrary loading.

The effect of an initial reaction force is included in the integral.

The slope $dw/dr(b_i)$ can be extracted from the first derivative of Eq. (4), that is

$$\frac{dw}{dr}(r) = \sum_{i=0}^{n} H(r, b_i)\left\{\frac{\alpha_i}{r}\int_{b_i}^{r} u \int^{u} \frac{1}{t}\int^{t} q(s) \times s\, ds\, dt\, du + \frac{1}{2}(\beta_i + C_1\alpha_i)\frac{r^2 - b_i^2}{r}\right\} + \frac{C_2}{r} \quad (5)$$

Successive substitution of $r = b_0, b_1 \ldots b_i$ into Eq. (5) leads to the expression

$$\frac{dw}{dr}(b_i) = \sum_{k=h+1}^{i}\sum_{h=0}^{i-1} \frac{\alpha_h\delta_{ki}}{b_k}\int_{b_i}^{b_k} u \int^{u}\frac{1}{t}\int^{t} q(s) \times s\, ds\, dt\, du + C_1 \sum_{k=h+1}^{i}\sum_{h=0}^{i-1} \alpha_h \frac{b_k^2 - b_h^2}{2\,b_k}\delta_{ki} + C_2 \sum_{k=1}^{i}\frac{\delta_{ki}}{b_k} \quad (6)$$

where

$$\sum_{k=r}^{h} \phi_r = 0 \qquad \text{if } h < r$$

$$\delta_{kk} = 1, \; \delta_{ki}\underset{i\neq k}{} = \sum_{s=k}^{i-1} \eta_s \frac{b_i^2 - b_s^2}{2\,b_i}\delta_{ks} + \sum_{s=k+1}^{i-1}\sum_{h=k}^{s-1} \alpha_s \times \frac{b_i^2 - b_s^2}{2\,b_i}\eta_h D_h \alpha_{kh}$$

with

$$\eta_s = (1 - \mu)\gamma_s/(b_sD_s)$$

Substitution of Eq. (6) into the deflection, slope, and moment equations, and a rearrangement in these equations of the constants of integration in terms of the initial parameters readily give the initial parameter functions and loading functions. The resulting initial parameter functions appear rather formidable, as indeed they are if a computer is not used for the solution of a plate with more than a dozen steps in D.

Once the initial parameter functions and loading functions have been determined, the solution of a stepped plate is no different from other initial parameter solutions. The initial parameters themselves are obtained by consulting a tabulation listing them for various boundary conditions in terms of the initial parameter functions and loading functions.[2] As is usual with the method of initial parameters, rather than using an integral form of these latter functions, a point change in deflection, slope, moment, or shearing force can be expressed by a "shift" of the parameter involved and its initial parameter function. Thus, the loading functions for a downward concentrated force of magnitude P at $r = a_j$ would be

$$W(r) = -Pw_4(r, a_j) \qquad \Theta(r) = -P\theta_4(r, a_j)$$

$$m(r) = -PM_4(r, a_j) \qquad v(r) = -PV_4(r, a_j)$$

Since several of the initial parameter functions do not remain finite as $b_0 = a_0 \rightarrow 0$, a solid (holeless or without inclusion) plate requires special attention. One of several methods of coping with this situation is as follows. Of the initial parameters, $\theta_0 = 0$, w_0 and V_0 have considerable useful physical significance, whereas M_0 has little. The w_0 terms remain finite. V_0 terms can usually be handled by retaining the appropriate initial parameters obtained as explained previously and permitting b_0 to remain small and nonzero until the initial parameters have been evaluated from the boundary conditions. The initial parameter M_0 can be replaced by $M_r(0)$, the moment per unit length, and then setting $C_2 = 0$ in Eq. (4) leads to properly adjusted values of initial parameter functions.

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Body Shape Effects on Skin Friction in Supersonic Flow

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Nomenclature

ν = kinematic viscosity
s = distance along body surface
x = axial distance
r_0 = local body radius
R = transformed body radius
δ = local boundary-layer thickness
δ^* = local boundary-layer displacement thickness
θ = local momentum thickness
H = local body shape parameter defined as δ^*/θ
$\bar{H}$ = dimensionless energy thickness
$H_{p,i}$ = flat-plate shape parameter
δ^{**} = energy thickness
C_f = local skin-friction coefficient
A = constant in momentum thickness equation
u = local velocity
U = transformed inviscid velocity
$U_{s,i}$ = transformed initial velocity
L = shape function
d = dissipation energy
t = turbulence energy
ξ = integrating variable
E = integrating variable
T^* = reference temperature
Re = Reynolds number

Subscripts

0 = stagnation value
e = value at the outer edge of the boundary layer
i = incompressible value
s = initial values
W = properties at the wall
∞ = freestream properties

THE calculation of the aerodynamic characteristics of aircraft and missiles at high Reynolds numbers requires an accurate knowledge of the contribution of the turbulent skin friction to the total vehicle drag. The skin friction drag C_F of an arbitrary shape exposed to fluid motion requires integrating the local wall shear stress τ_w over the body surface S:

$$C_F = \int_0^s \frac{\tau_w dS}{\frac{1}{2}\rho_\infty U_\infty^2 S} \tag{1}$$

In compressible axisymmetric flow τ_w is related to boundary layer and flow properties through, e.g., the Karman momentum integral equation

$$\frac{\tau_w}{\rho_e} u_e^2 = \frac{C_f}{2} = \frac{d\theta}{ds} + \theta\left[\left(\frac{2+H}{u_e}\right)\left(\frac{du_e}{ds}\right) + \frac{1}{\rho_e}\left(\frac{d\rho_e}{ds}\right) + \frac{1}{r_0}\left(\frac{dr_0}{ds}\right)\right] \tag{2}$$

A direct integration of Eq. (1) requires a knowledge of at least two of the three variables $\theta(s)$, $H(s)$, and $C_f(s)$. Unfortunately, no analytical expression relating pressure gradient to velocity profile, or local skin-friction coefficient to momentum thickness θ and shape parameter H is available. Because of these limitations, most of the methods[1–3] used to calculate turbulent boundary-layer characteristics over bodies with pressure gradients are approximate ones based on the integral forms of the momentum and energy equations in combination with empirically determined relations for skin-friction coefficient, such as the Ludwieg-Tillman relation[4] that includes the effect of shape parameter and boundary-layer velocity profiles. For incompressible flow, Truckenbrodt[5] gives for the momentum thickness θ_i,

$$\theta_i = \left(\frac{1}{U^3R}\right)\left[A\nu_0^{1/6}\int_0^x U^{10/3}R^{7/6}dx\right]^{6/7} \tag{3}$$

After subtracting the momentum equation from the kinetic energy equation and some rearrangements, he obtains also

$$\left(\frac{U_i\theta_i}{\nu_0}\right)^n(\theta_i)\frac{dL_i}{dx} = \left(\frac{U_i\theta_i}{\nu_0}\right)^n\left(\frac{\theta_i}{U_i}\right)\left(\frac{dU_i}{dx}\right) - K(L_i) \tag{4}$$

where

$$K(L_i) = \frac{-\left[2\dfrac{d+t}{U_i^2\rho_0} - H_i\dfrac{\tau_{w,i}}{U_i^2\rho_0}\right]\left[\dfrac{U_i\,\theta_i}{\nu_0}\right]^n}{(H_i-1)\,\bar{H}_i} \tag{5}$$

and

$$L_i = \int_{\bar{H}_{p,i}}^{\bar{H}_i} \frac{d\bar{H}_i}{(H_i-1)\bar{H}_i} \tag{6}$$

L_i is arbitrarily set equal to zero for zero pressure gradient flows; whence $H_{p,i}$ is equal to 1.4 for turbulent flows. An examination of existing turbulent flow data[4–8] led to the fol-

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